Mental Magnitudes and Increments of Mental Magnitudes

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Abstract: There is at present a lively debate in cognitive psychology concerning the origin of natural number concepts. At the center of this debate is the system of *mental magnitudes*, an innately given cognitive mechanism that represents cardinality and that performs a variety of arithmetical operations. Most participants in the debate argue that this system cannot be the sole source of natural number concepts, because they take it to represent cardinality approximately while natural number concepts are precise. In this paper, I argue that the claim that mental magnitudes represent cardinality approximately overlooks the distinction between a magnitude and the increments that compose to form that magnitude. While magnitudes do indeed represent cardinality approximately, they are composed of a precise number of increments. I argue further that learning the number words and the counting routine may allow one to mark in memory the number of increments that composed to form a magnitude, thereby creating a precise representation of cardinality.

Introduction

There is at present a lively debate in cognitive psychology concerning the origins of natural number concepts. At the center of this debate is the system of *mental magnitudes*, an innately given cognitive mechanism that represents cardinality and that performs a variety of arithmetical operations. Most participants in the debate argue that this system cannot be the sole source of natural number concepts, because they take it to represent cardinality approximately, while natural number concepts are precise. *Weak nativists* argue, however, that natural number concepts develop from the combination of mental magnitudes with other innate representations.[1] *Strong nativists* argue that natural number concepts cannot develop in this way, and that human beings innately possess concepts of at least the first few natural numbers.[2]

In this paper, I argue that the claim that mental magnitudes represent cardinality approximately overlooks the distinction between a magnitude and the increments that compose to form that magnitude. While magnitudes do indeed represent cardinality approximately, they

[1] This is not entirely accurate of all weak nativists. Susan Carey (2004, 2009a, 2009b) and Le Corre and Carey (2006, 2007) for example, argue that our acquisition of the first few natural number concepts depends on innate systems for representing objects and for understanding natural language quantifiers, that the system of mental magnitudes plays no role in this process, and that it only later plays a role in our acquisition of larger number concepts. I describe this position below.

[2] The use of the terms *strong* and *weak nativism* is due to Laurence and Margolis (2007).

are nevertheless composed of a precise number of increments. I argue further that learning the number words and the counting routine may allow one to mark in memory the number of increments that were composed to form a magnitude, thereby creating a precise representation of cardinality. The resulting hypothesis differs from strong nativist views in that it does not posit innate natural number concepts. It differs from weak nativist views in that it claims that the system of mental magnitudes is the only innate system to play a role in the acquisition of natural number concepts.[3]

The paper will have six main sections. In the first, I will review some of the experimental evidence that has led to the postulation of two innate representational systems. One is the system of mental magnitudes. The other is the system of *object-files*, which represents small numbers of discrete objects. In the second section I will describe an early attempt to explain the origin of natural number concepts in terms of mental magnitudes, and why that attempt has largely been rejected. In the third and fourth sections I will describe weak and strong nativist accounts, respectively. In the fifth section I will present the central argument of the paper: that because both strong and weak nativist accounts overlook the distinction between magnitudes and the increments that compose to form them, they overlook a way in which mental magnitudes may be the sole innate system to play a role in the acquisition of natural number concepts. In particular, mental magnitudes are composed of a precise number of increments, and learning the number words and the counting routine may allow one to mark in memory the number of increments in a magnitude, thereby creating a precise representation of cardinality. In the sixth section I will

[3] Saying that the present hypothesis appeals only to the system of mental magnitudes is somewhat inaccurate. In fact it will have to appeal to other resources, such as the ability to create one-to-one correspondences between representations. These other resources will be appealed to by other extant theories, however. Since my aim is to distinguish the present hypothesis from others, it is most useful to say, for example, that while some hypotheses appeal to object-files and mental magnitudes, and while others appeal to object-files and quantificational markers in natural language, the present hypothesis appeals only to mental magnitudes.

argue that the hypothesis offered here is not equivalent to any of the others discussed, and I will also discuss several objections and challenges that the hypothesis faces.

It is important to note that my intention here is primarily one of clearing logical space. That is, the main goal of the paper is to describe and explain a hypothesis that I believe has been overlooked in the literature, and to distinguish it from other extant hypotheses. The empirical evidence I present is aimed at serving those two goals, and not at arguing that the proposed hypothesis is on balance the best among the alternatives. While I present evidence that supports the proposed hypothesis, I will also acknowledge evidence that does not support it. To be sure, I will describe some unresolved challenges that it faces. Still, while all of the extant hypotheses face their own problems (as I discuss below), and thus the general debate remains unsettled, I take it that exploring alternative hypotheses remains a useful endeavor.

Before beginning, it is also important to describe some terminology and syntactic conventions I will use. First, let the natural numbers be the positive integers {1, 2, 3…}. This is a departure from some treatments of the naturals, which include the positive integers and zero {0, 1, 2…}, but the presentation here is simplified by referring to the positive integers as the natural numbers, and nothing substantive hangs on it. Second, when referring to the number of objects or events in a group (a cardinal number) I will use the appropriate number word (e.g., "There are fifty states in the U.S."). When referring to the place an object or event has in a list (an ordinal number) I will also use the appropriate number word (e.g., "George Washington was the first U.S. president. John Adams was the second."). When referring to particular numbers themselves, I will use the corresponding Arabic numeral (e.g., "2 is the unique successor of 1"). When referring to particular Arabic numerals or particular number words, I will use single quotes (e.g., "'2' and 'two' refer to 2"). When referring to particular natural number concepts, I will use small capitals (e.g., "A person with the concept TWO can enumerate sets with two items").

It is also imperative to begin by explaining what I mean by *natural number concepts*, and by the idea that natural number concepts are *precise*. Examples of natural number concepts are the concepts ONE, TWO, THREE…. These are individual concepts meaning 1, 2, 3… respectively, and must be distinguished from the general concept NATURAL NUMBER, meaning any n such that n is an element of {1, 2, 3…}.[4] The idea that number concepts are precise is the idea that a person in possession of a natural number concept, *N*, is among other things able to distinguish groups of objects with cardinality n from groups of objects with any other cardinality, including $n-1$ and $n+1$. For example, given enough time (and perhaps pen and paper) an adult in normal conditions will be able to determine that there are, say, exactly ninety-seven people in a large room.

I intend this as a necessary, and not sufficient, condition on possession of number concepts. I will not here attempt to provide a list of jointly sufficient conditions on possession of number concepts. Moreover, although I have phrased the requirement as behavioral, it restricts the class of representational systems that can count as natural number concepts. That is, any representational system that does not distinguish between the number n and its nearest neighbors cannot be a system of natural number concepts, because its employment could not explain the behavioral requirement. Finally, I want to stress the conjunctive nature of the requirement. Anyone who can distinguish groups of four objects from groups of three objects and groups of

[4] Rips, Asmuth, and Bloomfield (2006, 2008) and Rips, Bloomfield, and Asmuth (2008) discuss the origin of the general concept NATURAL NUMBER, and argue that it requires an at least implicit understanding of the axioms of arithmetic. But this is a separate problem from the problem of the origin of individual natural number concepts, and I will not take it up here. See Margolis and Laurence (2008) and Sarnecka (2008) for discussion.

five objects, but who cannot distinguish groups of five objects from groups of six objects, may possess the concept FOUR, but does not possess the concept FIVE.[5] [6]

1. Approximate Numerical Competence, Mental Magnitudes, and Object-files

Recent decades have seen an amassing of evidence (from a variety of looking-time methods[7]) that human infants possess a range numerical abilities, or in other words, that they are responsive to the number of objects in a set or sets.[8] In particular, it has been well established that infants are able to compare the cardinalities of sets of objects, and to compute sums and differences.[9]

For example, Xu and Spelke (2000) found that six-month-old infants habituated to displays containing eight dots (and controlled for variables besides cardinality) would dishabituate to displays containing sixteen dots (and when habituated to sixteen dots, would dishabituate to eight dots).[10] Similarly, Lipton and Spelke (2003) found that when habituated to eight sounds, six-month-olds would dishabituate to sixteen sounds (and again, vice versa). Moreover, McCrink and Wynn (2004) showed nine-month-old infants computer displays of five

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[5] This is important because, as Wynn (1992a) shows, children come to meet this requirement in stages, first developing the ability to distinguish groups of one from other groups, then groups of two, then groups of three, and then all at once groups of any size. I will describe this process in more detail below.

[6] Some may worry that the requirement as stated implies that we do not possess concepts for very large numbers, since we would not be able to distinguish them from their nearest neighbors. For example, one could not distinguish one billion trillion objects from one billion trillion plus one objects. But note that this inability is due to a lack of sufficient time and memory resources. In contrast, small children are unable to distinguish numbers from their nearest neighbors, even for very small numbers, so their inability cannot be due to time and memory constraints. Thus, the caveat might be added that the requirement is in fact not necessary for the possession of the concept *N*, if distinguishing *N* from its nearest neighbors would outstrip the time and memory resources available to the subject. It would remain unclear then, exactly what the necessary and sufficient conditions on possession of *N* would be. But that is not problematic for the present account—we do not, after all, possess the concept ONE BILLION TRILLION in exactly the same way that we possess small number concepts, and the present account is primarily concerned with the latter.

[7] See Carey and Spelke (1996) for a useful discussion of looking-time methods and results.

[8] There is also a great deal of research showing that many non-human animals also possess mathematical abilities. For early results see e.g., Mechner (1958), Platt and Johnson (1971), and Meck and Church (1983). For a recent review, see e.g., Cantlon, Platt, and Brannon (2009).

[9] Feigenson, Dehaene, and Spelke (2004) provide a useful review.

[10] The numbers involved here are important. For some evidence suggests that cardinalities from one to about three or four are represented not by mental magnitudes but by an object-tracking system. I will describe this system below.

objects being hidden behind a screen, to which five more objects were then added. The screen then dropped, revealing either five or ten objects. The infants who saw the mathematically incorrect outcome of five objects showed surprise (longer looking times) while the infants who saw the mathematically correct outcome of ten objects did not. McCrink and Wynn also reversed this design, starting with ten objects and removing five. Again, the infants who saw the mathematically incorrect outcome (ten objects) showed surprise, but the infants who saw the mathematically correct outcome (five objects) did not.

It is important to note that infants' abilities to distinguish sets of objects based on the cardinalities of the sets appears to be dependent on the relative cardinalities of the sets. For example, although six-month-olds distinguished between eight and sixteen objects, they failed to distinguish between eight and twelve objects (Xu and Spelke 2000). Similarly, while six-month-olds distinguish eight from sixteen sounds, they fail to distinguish eight from twelve sounds (Lipton and Spelke 2003).[11] [12] This implies that infants' numerical abilities conform to Weber's Law, which states that whether or not a subject can discriminate two stimuli depends not on the absolute values of the stimuli, but on the ratio between the two.

It is also important to note that early studies did not successfully show that infants were in fact responding to cardinality. In particular, some studies used small numbers of objects and thus left open the possibility that infants were tracking discrete physical objects, a strategy that would implicitly provide correct arithmetical solutions.[13] Also, other continuous variables

[11] Some early experiments revealed this pattern too, as both Starkey and Cooper (1980) and Antell and Keating (1983) found that infants failed to discriminate four from six dots, and Strauss and Curtis (1981) found that infants were unable to distinguish between four and five dots. Strauss and Curtis (1981) also tested infants at three versus four, a discrimination which female infants were able to make and male infants were not. But again, these early studies failed to control for variables besides number (see footnote 14).

[12] Notice, however, that this sensitivity does appear to improve with age. While Lipton and Spelke (2003) found that six-month-olds were unable to distinguish between eight and twelve sounds, they found that nine-month-olds were successful at discriminating eight from twelve sounds, though they failed to discriminate eight from ten.

[13] For example, Wynn (1992b) used small numbers of toys.

besides cardinality, such as total brightness of the displays or total surface area of the objects in the displays, were not controlled.[14] Thus it was possible that infants were responding to these other variables instead of cardinality. More recent work suggests that when small numbers of objects are involved, infants' numerical abilities are subject to a set size limit of about three or four, rather than to Weber's Law. [15] Taken together these studies suggest that when small numbers are represented, infants may rely on a system that tracks small numbers of discrete physical objects (or events), and that they may in some instances respond to variables besides number, such as total contour length of the objects in a display.

So it is now widely accepted[16] that numerical competence in infants is to be explained by two innate representational systems (FIGURE 1). Many authors now believe that in the small number range (one to about three or four objects or events) a system of *object-files* explains numerical competence. Such a system tracks discrete physical objects or events in an imagistic format that includes various properties of those objects such as size and shape. This system does not represent cardinality *per se*, but does implicitly represent cardinality in the sense that *n* objects are represented via *n* object-files, and therefore supports arithmetic reasoning.

In the large number range (more than three or four objects or events) a system of mental magnitudes is responsible for numerical competence. Such a system represents numbers as magnitude values that vary in size, with the variation increasing as the numbers represented increase. Sometimes termed an *accumulator*, the model was originally proposed by Meck and Church (1983) to explain numerical competencies in rats. Because discussion of this system will

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[14] For instance, Starkey and Cooper (1980) used displays containing arrays of dots, with dots of equal size and spacing, such that an array with six dots would be longer and contain more total dot surface area than an array with four dots. See also Strauss and Curtis (1981).

[15] See for example Clearfield and Mix (1999, 2001) and Feigenson, Carey, and Spelke (2002).

[16] Though not entirely – see for example Gallistel, Gelman, and Cordes (2006) and Cordes and Brannon (2008a, 2008b) for debate.

take a central role in the rest of the paper, I will take the next few paragraphs to describe it in detail.

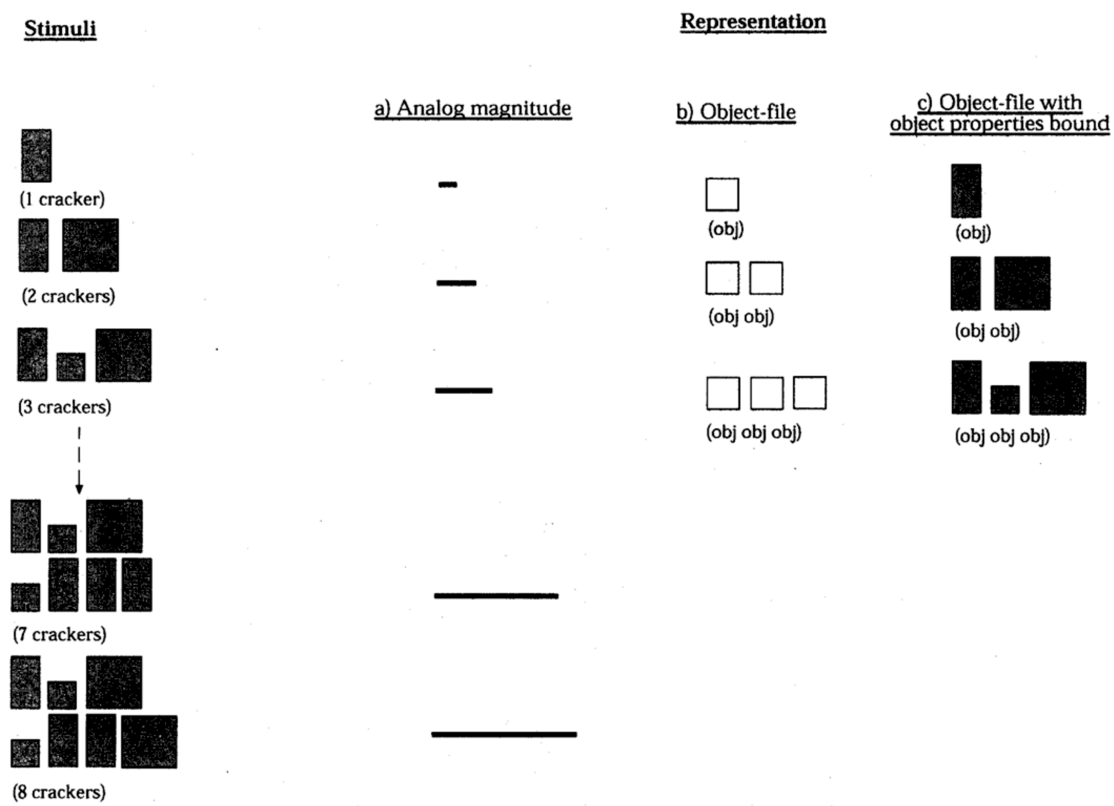

Figure 1. Stimuli and various representational formats of those stimuli (Feigenson, Carey, and Hauser 2002).

Meck and Church (1983) originally described the system as having three parts: a *pacemaker*, a *switch*, and several *basins*. The pacemaker creates pulses at a (somewhat) constant rate. When the switch is closed, these pulses are transferred into a basin, where they are stored. The switch may close and open again at periodic intervals, thus increasing in steady increments the pulses that are passed into the basin. If *n* objects are observed, *n* increments may be passed from the pacemaker to the basin.[17]

[17] Some authors hold that it can also be used to measure duration. See Meck and Church (1983). Also see Gallistel, Gelman, and Cordes (2006).

This is a system of *magnitude values*: representations are distinguished by their size. For comparison, consider the Arabic numerals. Here the size of a numeral has nothing whatsoever to do with what it represents. For example "2" and "2" both represent the same number. Moreover, the accumulator is a *proportional* system: representations grow in proportion to the number represented. For comparison, consider again the Arabic numerals, and their common usage as a base ten system. These representations do not grow in proportion to the number represented.[18] [19]

However, the accumulator's increments are inherently variable, in the sense that any two increments are only roughly equivalent in magnitude. The result is that two representations of the same number may differ in magnitude. That is, numbers are denoted not by particular magnitudes, but by ranges of magnitudes. In other words, it is not strictly true that if two representations denote the same number, they have the same magnitude.

Finally, the variability inherent in individual increments compounds, such that representations denoting larger numbers exhibit more variability than do representations denoting smaller numbers. More exactly, the standard deviation of the sizes of magnitudes representing a given number is a linear function of the size of that number.[20] This feature of the system is known as *scalar variability*.[21] An effect of scalar variability is that, at a certain point, the range of magnitudes which may denote a given number becomes large enough that there is

[18] They grow, in the sense of requiring more digits, (roughly) in proportion to the logarithm of the number represented.
[19] Note that there are non-proportional systems of magnitude values. For example, suppose I have cups of water, each containing a number of ounces of water (from zero to nine ounces). But rather than use the empty cup to represent 0, the cup with one ounce to represent 1, and so on, I assign numbers to cups randomly so that e.g., cups with three ounces of water represent 7, cups with two ounces of water represent 9, etc.
[20] As LeCorre and Carey (2007) put it, "Scalar variability holds the standard deviation of the estimate of some quantity is a linear function of its absolute value" (397).
[21] Note that I assume scalar variability is caused by the compounding of noise in the size in the individual increments. Some may worry that this assumption is unfounded, as scalar variability could also be achieved with increments that were not noisy in size, but if there was a probability that for each object observed, the system produced between zero and two increments. But physical systems are inherently noisy, so there is already a reasonable explanation of scalar variability without needing to appeal to too many or too few increments. That hypothesis needlessly complicates matters.

overlap with the ranges of magnitudes that may denote other nearby numbers. Thus two representations with the same magnitude may differ in content.[22] In other words, it is not strictly true that if two representations have the same magnitude, they denote the same number. [23] [24]

An accumulator can be used to compare sets of objects by cardinality, by filling one basin for each set, and comparing the levels of fullness of the basins. It can be used for addition, by combining the contents of two or more basins. It can also be used for subtraction, multiplication, division, and so on.[25] Wynn (1992c) gives a clear and detailed argument in favor of using the model to explain numerical competencies in infants. Moreover, the model provides an especially

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[22] A possible objection here that it is inappropriate to say that two magnitudes that are equal in size may represent different numbers, and that it would be more correct to say that some magnitudes represent a range of numbers. However, I take the representational content of a magnitude to be determined by the number of objects in response to which it was formed. A magnitude represents, for example, the number 5 just in case it was formed when the subject observed five objects or events.

[23] As an illustration of the accumulator, many authors have described a model accumulator constructed using a supply of water and cups. Such a model provides an intuitive account of the variability inherent in accumulator representations, and indeed, of why the accumulator exhibits scalar variability. For instance, Dehaene (1997) writes,

> A clear drawback of the accumulator is that numbers, although they form a discrete set, are represented by a continuous variable: water level. Given that all physical systems are inherently variable, the same number may be represented, at different times, by different amounts of water in the [basin]. Let us suppose, for instance, that water flow is not perfectly constant and varies randomly between 4 and 6 liters per second, with a mean of 5 liters per second. If [the user] diverts water for two-tenths of a second into the [basin], one liter on average will be transferred. However, this quantity will vary from 0.8 to 1.2 liters. Thus, if five items are counted, the final water level will vary between 4 and 6 liters. Given that the very same levels could have been reached if four or six items had been counted, [the] calculator is unable to reliably discriminate the numbers 4, 5, and 6. (29-30)

[24] In the above description of the accumulator I have omitted three areas of disagreement. First, Meck and Church's (1983) model enumerates objects serially. Other models (e.g., Dehaene and Changeuax 1993) enumerate objects in parallel. Second, as described above there is variability inherently present in the model. Other descriptions (e.g., Gallistel, Gelman, and Cordes 2006) take the variability to be present in memory, not in the accumulator itself. Third, as explained above, according to this model the system's inability to distinguish between numbers that are close together is due to scalar variability. In other models (e.g., Dehaene, Izard, Spelke, and Pica 2008 and see also Cantlon, Cordes, Libertus, and Brannon 2009 and Dehaene, Izard, Pica, and Spelke 2009) this inability is due to a logarithmic compression of increment size. The second and third of these disagreements should have no effect on the arguments in the rest of this paper. The first, however, is relevant. The view I present below depends on the accumulator adding increments serially, as the subject slowly observes a group of objects (e.g., observes the objects one at a time). But the present view does not depend on the accumulator only operating in this manner; it allows that the accumulator may also operate in parallel. I will address this question in more detail in the concluding section. Note though that whether it operates serially or in parallel, or both, remains an open question. So it is worth exploring what sorts of hypotheses are available, assuming particular answers to that question. The present argument is that there is an important hypothesis that has been overlooked, assuming that mental magnitudes can be formed serially.

[25] Gallistel et al. (2006) argue that some animals compare rates of return from foraging in different areas, and that positing an accumulator can explain this as well.

powerful explanation of why infants' numerical abilities are subject to Weber's Law. In particular, because the system exhibits scalar variability, it cannot reliably distinguish between nearby cardinalities, where what counts as "nearby" is proportional to the size of the given cardinalities.

Recall, however, the requirement for possession of natural number concepts described above: that one be able to distinguish between a given number n and its nearest neighbors $n+1$ and $n-1$, regardless of the size of n. Because the accumulator exhibits scalar variability, it does not distinguish between a given number and its nearest neighbors, and therefore it cannot underlie this ability. In other words, the feature of the accumulator that makes it so powerful an explanation of numerical competencies in infants also rules it out as a candidate for providing natural number concepts.

Still, because the content of mental magnitudes are numerical (the accumulator responds to cardinality, after all) it is natural to hypothesize that the system plays a role in the development of mature numerical thought. But consider the details of the problem. Infants possess an innate representational system that allows them to distinguish between groups of objects based on the cardinality of those objects, but only if there is sufficient difference between those cardinalities. Possession of natural number concepts, however, requires a representational system that allows one to distinguish between groups of objects based on cardinality even when the cardinalities differ by one, regardless of the absolute size of the cardinalities. How, if at all, can the former give rise to, or help to give rise to, the latter?

Authors have generally described this problem in terms of the problem of learning specific number concepts. For example, a child does not have the concept SEVEN until she understands that that concept (and the accompanying number word and symbol) refers to an exact cardinality (i.e., that the concept no longer applies to a group to which it formerly applied, when an item is taken away or another is added). How do children acquire precise representations of number, given a system of representation that does not distinguish between such nearby cardinalities?

At present, there are two kinds of answers to this question. The first says that children solve this problem by making use not only of the system of mental magnitudes, but also of the system of object-files. The second says that the problem cannot be solved, and that humans must therefore be endowed with an innate system that represents numbers precisely. Below, I will describe both approaches. I will begin however, by presenting an early attempt to solve the problem by appeal only to the accumulator, an attempt that has since been abandoned.

2. From Mental Magnitudes to Weak Nativism

Some authors have in fact argued that the system of mental magnitudes can explain the development of precise number concepts (though as I will explain below, they have since backed away from that claim). For instance Gallistel, Gelman, and Cordes (2006) suggest that the process of learning the number words may allow a child to "pick out" precise representations from her innate approximate ones. In particular, they claim that the accumulator's representations are homomorphic to the so-called *counting principles*, described in Gelman and Gallistel (1978). Those principles describe what a child must be able to do, before she can be said to have mastered the counting routine, using a particular set of number symbols (words, numerals, etc.).

There are three principles.[26] The first is the *one-one principle*: each symbol in the set may be used only once in an episode of counting. The second is the *stable-order principle*: the symbols must be used in a fixed order, that is, they must be used in the same order in all episodes of counting. The third is the *cardinality principle*: the final symbol used in a given count represents the cardinality of the set of objects counted.

The idea in Gallistel et al. (2006) is that the system of mental magnitudes operates in accord with these principles, as do number words when used appropriately. Thus, when we count, we use each count word only once, we use them in a fixed order, and the last word in the count indicates the cardinality of the set counted. Gallistel et al. claim that,

> Each count in the nonverbal process defines a next magnitude. Thus… the magnitude that results from the next step is always one fixed increment greater than the magnitude that results from the previous step… Finally… the magnitude produced by the final increment in the nonverbal counting process [represents the cardinality of the set being counted]. (2006, 265)

And they claim that,

> …the child perceives the homomorphism between the nonverbal and the verbal counting process, and this leads to the assumption that the words used in the counting process represent the same aspect of the world as do the mental magnitudes obtained from the nonverbal counting process… [This] means that the child thinks that the counting words refer to the same things in the world as the mental magnitudes representing [cardinality] refer to…. (265-6)

Thus, Gallistel et al.'s proposed solution is that the process of learning to use number words and symbols is the process of associating those words and symbols with mental magnitudes, such that precise representations of number are generated or picked out from mental magnitudes.[27]

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[26] Gelman and Gallistel (1978) also describe the *abstraction principle*: that the symbols may be used to count heterogeneous groups of objects (e.g., toys together with cookies), and the *order-irrelevance principle*: that the order in which the objects are counted does not affect the cardinality of the group. Neither of these will affect the discussion here though, so I leave them out.

[27] Gallistel at al. (2006) are not alone in this line of thinking. Dehaene (1997) describes the process in similar ways. He explains that to learn the meaning of the word "three," for example, the child must

It is necessary to note here that Gallistel et al. take mental magnitudes to be a continuous form of representation. They write that, "scalar variability suggests that [cardinality] is represented… by mental magnitudes, that is, by real numbers, rather than by discrete symbols like words or bit patterns" (252). Moreover, they say explicitly that, "our underlying representation of [cardinality] has the continuous character of the real numbers rather than the discrete character of integers" (268). But there is tension between this view and their suggestion about how children acquire natural number concepts.

They write that their hypothesis,

> raises interesting questions about where some of our intuitive convictions about quantity come from. One of these concerns our concept of exact equivalence. Empirically, there is no such thing as exact equivalence among uncountable quantities… If we relied simply on our experience of uncountable quantities, we would not have a concept of exact equivalence…
>
> Nonetheless, we believe that when equals are added to equals, the results are equal. We believe this despite the fact that it may or may not be true of mental magnitudes, depending on whether the nonverbal system for reasoning with magnitudes recognizes equivalence… and how it decides whether two magnitudes are equivalent. (2006, 266)

Gallistel et al. present several possibilities for how this problem might be solved, though they are speculative.[28]

Laurence and Margolis (2005), however, argue that the idea that mental magnitudes alone provide the origin of natural number concepts cannot be true, that none of Gallistel et al.'s

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> correlate [his] preverbal representation with the words he hears. After a few weeks or months, he should realize that the word 'three'… is very often mentioned when his mental accumulator is in a particular state that accompanies the presence of three items. Thus, correlations between number words and his prior nonverbal numerical representations can help him determine that 'three' means 3. (107)

De Cruz (2008) writes that "Natural language is one among several tools that allow us to map exact cardinalities onto our approximate… mental number [representations]" though she also says that "External symbolic representations of natural numbers are not merely converted into an inner code; they remain an important and irreducible part of our numerical cognition" (487).

[28] For instance, they suggest that the system may judge two magnitudes to be equivalent if it cannot reliably order them, or alternatively that the system may employ "shortcuts" such as assuming that when one increment is added to each of two equivalent magnitudes the results are equivalent, or finally, that "the discrete nature of the verbal representation… is the origin of our notion of exact equivalence" (Gallistel, et al. 2006, 266-7).

proposed solutions can work, in particular because mental magnitudes are not in fact homomorphic with the counting routine. They explain that mental magnitudes do not increment in a way that defines successor increments. Thus they write that,

> Assuming that the Accumulator's states do represent the reals, it's hard to see how the Accumulator could embody the counting principles. The idea that there is a 'next tag' makes no sense with respect to the reals. The problem is that the reals are *dense* in that between any two real numbers there is always another real number. So '2' is no more 'the next tag' after '1' than '1.5' is (or for that matter, than any other number greater than 1 is). (224)

And second, they argue that,

> …even if there was some sense in which 'the next tag' could be defined for a system representing the reals, the Accumulator would still have to operate with impossibly perfect precision to ensure that the same [basin] levels are applied in the same order for each count. In all likelihood the level corresponding to '1' would rarely be followed by the level corresponding to '2'; rather, it would sometimes be followed by '2.0000000000103,' sometimes by '2.000010021,' and so on. But that's just to say that the stable-order principle wouldn't hold. (224)

In later papers, Gallistel and Gelman have agreed that this problem is fatal for the idea that the accumulator is the sole innate origin of natural number concepts.[29] For instance Leslie, Gelman, and Gallistel (2008) write that,

> One use we make of integers is counting things. A fundamental intuition here is that if three things are counted, then the resulting cardinal value will be exactly equal to the cardinal value that will result from counting them again. It is hard to account for this intuition if the brain represents cardinal values by noisy reals [i.e., by mental magnitudes] because two exactly equal values will never occur. Although we find compelling evidence for the existence of an analogue magnitude representation underlying counting and other number tasks, exact equality challenges such models. We are left without an account of why our basic number concepts—the ones picked out by language—should be integers rather than reals…. Two real-valued measures of the same entity are, in general, infinitesimally likely to be exactly equal, and infinitesimally likely to have an integer value. Thus, the chance of a child entertaining an integer hypothesis would be infinitesimal. If no child would learn integer values, then no language would contain words for such values; yet both are commonplace. Exact equality is an

[29] Carey (2001) argues on similar grounds that mental magnitudes cannot be the sole source innate source of natural number concepts.

> important constraint on representations underlying natural number concepts. (213-14)

They argue, therefore, that children must have innate natural number concepts, explaining, "We argue that basic number representation in humans is not limited to the reals; it must include a representation of the natural numbers *qua* integers" (215). Laurence and Margolis (2007)[30] also argue that human beings must possess innate natural number concepts, though their account differs from Leslie et al.'s. I will describe them both in some detail below, but these strongly nativist accounts are not the only proposed solutions to the problem. Elizabeth Spelke and Susan Carey, for instance, have each offered weakly nativist accounts, according to which natural number concepts are arrived at by combining mental magnitude representations with other innate representations. I turn now to a discussion of these solutions.

3. Weak Nativism

Recall that there is now wide agreement that numerical competence in infants is to be explained by two systems. One is the accumulator, representing numbers of objects greater than about three or four. The other is the system of object-files, representing one to about three or four distinct physical objects, and representing cardinality only implicitly. For weak nativists, the solution to the problem described above is that while neither system is alone capable of providing natural number concepts, together they are.[31]

As Spelke (2003) explains the problem,

> [The system of object-files] represents small numbers of persisting, numerically distinct individuals exactly, and takes account of the operations of adding or removing one individual from the scene. It … does not permit infants to

[30] See also Margolis and Laurence (2008).

[31] Carey (2004, 2009a, 2009b) for instance, describes acquisition of natural number concepts as a process of Quinean *bootstrapping* – the combination of innately given representations in ways that allow for new conceptual representations. See Quine (1960).

> discriminate between different sets of individuals with respect to their cardinal values. [The system of mental magnitudes] represents large numbers of objects or events as sets with cardinal values, and it allows for numerical comparison across sets. This system, however, fails to represent sets exactly… and therefore it fails to capture the numerical operations of adding or subtracting one. (2003, 299)

In other words, neither system does what is needed: explicitly represent the precise cardinalities of sets of objects. Though object-files represent cardinality precisely, they do so only for small groups of objects, and they do so only implicitly, insofar as they require *n* object-files to represent *n* objects. Mental magnitudes explicitly represent cardinalities of larger groups of objects, but only approximately. What is needed is a system that explicitly represents cardinalities, both small and large, precisely.

However, Spelke sees the fact that there are two systems at work as the solution to the problem. In particular, she argues that the explicit representation of precise cardinal values results from the combination of the precision inherent in the object-file system with the explicit representation of cardinal value inherent in mental magnitudes. That combination, moreover, occurs over a period of time as children learn to appropriately use the number words to identify cardinalities.

Following Wynn (1990, 1992a), Spelke explains that children generally learn the counting routine, 'one, two, three...' well before they learn to apply these words appropriately. She notes that there is a four-stage process during which children learn to use the counting words correctly. During the first stage, children appropriately use the word 'one' to refer to single objects, and they use number words besides 'one' to refer to more than one object, but they fail to distinguish between these other words. For example, when asked to point to a group of two, or a group of three fish, they will point randomly at either group. In the literature, such children have come to be known as "1-knowers", because they possess the concept ONE, but no other number concepts.[32]

During the second stage, children begin to appropriately use the word 'two'. They will pick two items from a group when asked to, and they will pick more than two items when asked for other numbers of items. However, they will not distinguish between number words other than 'one' or 'two'. If asked for a number of items greater than two, that is, they will pick numbers of items at random. Such children are described as "2-knowers". Children similarly come to learn to use the word 'three' ("3-knowers") and finally, they come to use words for numbers larger than three as a group (they begin to use them appropriately all at the same time).[33] These children are described as "*n*-knowers".

The process takes about a year to a year and a half, and Spelke suggests that it takes such a long time precisely because there are two mechanisms at work. In particular, she thinks that the word 'one' is learned readily because it only represents an individual object. That is, children learn the meaning of 'one' by "relating this word to representations constructed by their… system for representing objects" (2003, 301). Other numbers, however, require children to employ both the system of object-files and the accumulator. She writes,

> To learn the full meaning of *two*… children must combine their representations of individuals and sets: they must learn that *two* applies just in case the array contains a set composed of an individual, of another, numerically distinct individual, and of no further individuals. The lexical item *two* is learned slowly, on this view, because it must be mapped simultaneously to representations from two distinct… domains. (2003, 301)

[32] Note the relationship here to the necessary condition on possession of individual number concepts I gave above. The requirement is that a person in possession of the concept N must be able to distinguish groups of objects with cardinality n from groups of objects with cardinality $n+1$ and from groups of objects with cardinality $n-1$. The intuition that 1-knowers possess the concept ONE, but no other number concepts, is the source of this requirement.
[33] See Spelke (2003, 299-301).

Eventually, children learn the meanings of 'two' and 'three', by mapping the words to both systems. To learn the rest of the counting words, however, Spelke suggests that children perform an induction from the pattern observed. That is, children may notice that, "the progression from *two* to *three* in the counting routine is marked by *the addition of one individual to the set* [and also by] *an increase in the cardinal value of the set*" (302). Having noticed these facts, they may then "generalize these discoveries to all other steps in the counting routine." That is, they may,

> realize that every step in the counting routine is marked by the successive addition of one individual so as to increment the cardinal value of the set of individuals. Because these representations exceed the limits of [both object-files and mental magnitudes], these realizations depend on elaborate conceptual combinations. Those combinations, in turn, may depend on the natural language of number words and of the counting routine. (2003, 302-3)

In short, learning the correct use of number words takes several years because it is the process whereby number words become associated with both object-files and mental magnitudes. But while it is a slow process, it provides a solution to the problem. That is, mental magnitudes and object-files become combined, and as such come to be able to explicitly represent precise cardinal values.[34]

Spelke is not the only theorist to argue for a solution that does not involve innate natural number concepts. Susan Carey does as well. One of the most interesting facets of Carey's approach though, is that mental magnitudes play no role in the initial development of natural number concepts. Instead, natural number concepts first arise only as small number concepts, provided by object-file representations, but "enriched" by quantificational markers in natural language.

Carey explains that the object-file system,

[34] Condry and Spelke (2008) put the hypothesis this way: "the counting routines of specific human cultures engender spontaneous, constructive processes within the child and that these processes build a unitary system of natural number concepts from a set of conceptual primitives delivered by distinct, core cognitive systems" (37).

> creates working-memory models of sets. The symbols in these models represent particular individuals—this box, which is different from that one. However… even when drawing on [object-files] alone, infants have the capacity to represent two models and compare them on the basis on 1-1 correspondence. For representations of this format to subserve the meanings of the singular determiner or the numeral 'one' for subset-knowers [that is, 1-knowers, 2-knowers, or 3-knowers], the child may create a long-term memory model of a set of one individual and map it to the linguistic expression 'a' or 'one'. Similarly, a long-term memory model of a set of two individuals could be created and mapped to the linguistic expression for a dual marker or 'two', and so on for 'three' and 'four'… What makes these models represent 'one', 'two', and so forth is their computational role. They are deployed in assigning numerals to sets as follows: The child makes a working-memory model of a particular set he or she wants to quantify… He or she then searches the models in long-term memory to find that which can be put in 1-1 correspondence with this working-memory model, retrieving the quantifier that has been mapped to that model. (2009a, 248-9)

Once the child has these long-term memory models, Carey thinks, he is in a position to notice that whenever a set is accompanied by the word 'two', the recited counting routine is 'one, two', and whenever a set is accompanied by the word 'three,' the recited counting routine is 'one, two, three', etc. In other words, he is in a position to "notice that for these words, at least, the last word reached in a count refers to the cardinal value of the whole set" (2009a, 250). After noticing this, Carey thinks that the child may "notice an analogy between next in the numeral list and next in the series of mental models… related by adding an individual," and that he is now in a position to make the "crucial induction" that "if '*x*' is followed by '*y*' in the counting sequence, adding an individual to a set with cardinal value *x* results in a set with cardinal value *y*" (2009a, 250).

Thus, Carey's view is that the process of acquiring natural number concepts involves first creating long-term memory models of sets of individuals and mapping the number words to those models. Second, it involves noticing that the last word in an episode of counting represents the cardinality of the set counted, and finally, it involves the induction that for any subsequent word in the count list, it must refer to a subsequent cardinality. Again, one of the most interesting aspects of this account is that it does not involve the system of mental magnitudes. Carey acknowledges though, that "a further bootstrapping episode… integrates the numeral list with analog magnitude number representation, greatly enriching their numerical content" (2009a, 251). Still, Carey admits that it is a "surprising upshot" that "one of the evolutionarily ancient systems of representation with numerical content, the [mental] magnitude system, plays no role in providing initial meaning for verbal numerals" (2009a, 251).

Some authors[35] have complained that Carey's account cannot explain the origins of natural number concepts, since the content of the representations is not numerical. As with object-files (without enrichment), they only implicitly represent numbers. There are also problems for weak nativism in general, stemming from the idea that object-files and mental magnitudes can combine to form new, meaningful representations. For instance, Gallistel et al. (2006) worry that if the ability to add has its origins in the system of object-files, then it will not support the arithmetic closure principle,

> because there will be no symbol to represent the results of adding 'threeness to threeness'. Thus, if operations with these very limited sets of mental symbols are the foundation of numerical understanding, it is a puzzle how we come to believe in the infinite extensibility of number, in the fact that you can always add one more". (269-70)

Moreover, it remains unclear exactly how object-files and mental magnitudes combine and what the resulting representations are like. Gallistel et al. (2006) write that,

> The two systems would seem to be immiscible for the same reasons that analog and digital computers cannot be hybridized. Although both do arithmetic, they do it in fundamentally different ways. Thus, there is no way of adding a digitally represented magnitude (for example, a bit pattern) to a magnitude represented by an analogical magnitude (for example, a voltage), because the two forms of representation are immiscible. It is hard to see why this same problem does not arise in the developing human mind, if it represents some numbers discretely and others by means of mental magnitudes. If oneness is represented discretely but

[35] See for example, Montemayor and Balci (2007).

> tenness is represented by a mental magnitude, how is it possible to mentally add oneness and tenness? (270)

And Gelman and Gallistel (2004) ask the related question,

> If the brain represents [for example] three and seven in fundamentally different ways, how can it compose them arithmetically (order them, add them, etc.)? What representation form do the resulting hybrids have? This is particularly puzzling when two numbers beyond the discrete and precise range are subtracted to yield a number inside it, as in 7 – 5 = 2. (442)[36]

Because of the difficulties for weak nativist positions, and because of the difficulties for explaining natural number concepts solely in terms of mental magnitudes, several authors have argued for strongly nativist positions. In the next section I will describe two such accounts.[37]

4. Strong Nativism

According to strong nativists, human beings possess innate concepts of at least some natural numbers. Leslie, Gelman, and Gallistel (2008) for instance, suggest that humans possess an innately given representation of the number one (i.e., the concept ONE),[38] and an innately given recursive rule for producing the successor of any representation of a natural number. Hence, all natural number concepts can be produced. They also posit an innate inference rule guaranteeing the multiplicative identity of the concept ONE. This rule is to account for the fact that the concept ONE is special among natural number concepts, in that when any number *n* is multiplied by 1, the

[36] Also see Laurence and Margolis (2005). Rips, Asmuth, and Bloomfield (2006, 2008) and Rips, Bloomfield, and Asmuth (2008) have argued against both forms of weak nativism described here, on grounds that the induction required cannot guarantee natural number concepts. The reason is that it cannot rule out other non-standard conceptual structures, such as loops. As Margolis and Laurence (2008) have pointed out though, this appears to be an instance of more general worries about induction (such as those discussed by Kripke 1982) and would seem to affect any account of the acquisition of number concepts, including strongly nativist accounts. The only exception would be an account according to which the entire set of natural number concepts is innate, but such an account would appear absurd on its face, and no one has offered such an account.

[37] There are other weakly nativist accounts, in addition to Spelke's and Carey's. For instance, Decock (2008) discusses the possibility that number competence develops from the one-to-one principle, rather than from enumeration.

[38] Though they note the possibility that the innate complement of natural number concepts could include the first few numbers.

result is *n*.[39] Moreover, they suggest that each innate or generated natural number representation is associated with a mental magnitude value. This is simply in keeping with the experimental evidence that links use of number words and symbols with deployment of mental magnitudes.[40]

Leslie et al. note that these innate and generated representations could be akin to mental hashmarks, but that such a system is limited in its usefulness, as any system is whose symbols grow in proportion to the numbers they denote. For example, it would be impossible to consider the solution to 1,809,672 x 3,432,864 if this were to be computed in hashmarks. Thus they also posit a "compact notation," in which representations grow in proportion to the logarithm of the number represented, and in which each representation is associated with a unique hashmark representation. They allow that the compact notation itself may be innate or it may be acquired, perhaps from natural language itself, since number words and common number symbols do not grow in size in proportion to the numbers they represent.

Leslie et al. are not alone is positing innate representations of natural numbers. Margolis and Laurence (2008) write that mental magnitudes "are by their nature approximate and hence incapable of expressing a difference of exactly one" (935) and they argue in favor of an innate "number module" that represents the first three or four integers. Laurence and Margolis (2007) construe these representations as having "precise numerical content," but a minimal amount. For instance, they leave it open whether the system possesses any understanding of the mathematical relations that hold among the numbers it represents, including whether they are ordered. "What makes [these representations] numerical," they write, is just that they serve to detect collections

[39] Leslie et al. (2008) claim that any candidate for being the concept ONE must have this feature. But it is not clear that all features of the concept ONE must be innate. The idea that the number 1 is the (unique) multiplicative identity could be acquired as multiplication is learned. And this is true regardless of whether the concept ONE is innate or not.

[40] The experimental evidence shows that when estimating the cardinality of a group of objects, adults are able to produce a number word, and adults' rapid discrimination of Arabic numerals exhibits the well-known "distance" and "magnitude" effects, which are corollaries of Weber's Law, and hence suggests the deployment of mental magnitudes. For further explanation and discussion, see for example Dehaene (1997) and Gallistel et al. (2006).

of specific sizes, for example, the representation corresponding to 2 is uniquely responsive to collections that have precisely two items, independent of whatever non-numerical properties the collections have" (146).

Since Margolis and Laurence do not posit any innate way of generating integers beyond the first four (e.g., something akin to Leslie et al.'s recursive rule for generating successors), it remains a question how children acquire concepts of the natural numbers beyond the first few. They suggest the possibility that an "external structured symbol system" such as natural language plays an important role. For instance, children might map the words 'one', 'two', and 'three' directly to their innate representations of those numbers provided by the number module. They may learn the counting routine as a game, but if they notice that the last word in the count sequence expresses the number of objects in the group counted, that may provide an understanding of the ordering of their innate representations. Also, their representation of the number 1 may allow them to notice that the numerical difference between any two number words in the count list is one. Finally, they may be able to perform an inductive inference concluding that for each successive number word in the counting sequence, that successive number word refers to a number one greater than the last.

The argument for (some version of) strong nativism rests largely on the idea that none of the weak nativist positions described above are sufficient to explain the origin of natural number concepts. Since neither mental magnitudes alone, nor mental magnitudes in conjunction with object-files can supply natural number concepts, yet humans obviously possess these concepts, we must possess them innately. Of course, weak nativist proposals themselves serve as arguments against strong nativism. From the weak nativist point of view, strong nativism is simply not necessary, since number concepts can be accounted for in virtue of mental magnitudes. Moreover, work with some indigenous groups of people suggests that concepts of natural numbers are not innate. Thus, the Pirahã of Brazil lack words in their language for numbers above about two (roughly, they possess a "one, two, many" language), and by some accounts they do not possess even very small natural number concepts (Gordon 2004, but also see Laurence and Margolis 2007 and Frank et al. 2008).[41] [42]

It is important to recall here that what drives all these views, both strongly and weakly nativist, is the idea that mental magnitudes represent cardinality in an approximate manner, and therefore that they cannot alone be the source of natural number concepts, since the latter are precise. Weak nativists attempt to solve the problem by appeal to conceptual combinations, strong nativists to innate natural number concepts. However, I think there is an important distinction overlooked in the literature. The distinction is between a magnitude and the increments that compose to form that magnitude. In the remainder, I will argue that attention to this distinction may provide a new solution to the problem.

5. Magnitudes and Increments of Magnitudes

Consider three walls. The first is constructed of bricks, joined using mortar that is a different color than the bricks. When the wall is complete, it is obvious that it was formed by combining parts, and it is obvious where each of those individual parts begins and ends. The second wall is constructed of blocks of ice, which are laid directly against and on top of one another. The blocks of ice gradually melt together, such that the wall no longer appears to have

[41] Also see Izard, Pica, Spelke, and Dehaene (2008), for discussion of speakers of Mundurucú, who possess numerical terms that "approximately" correspond with the numbers one through five.
[42] The argument from the claim that some cultures do not possess natural number concepts to the claim that those concepts cannot be innate depends on a notion of innateness according to which if a concept is not present in all cultures then it is not innate. Such an account would have to be defended if the argument were to be successful. I am here noting that some have made the argument, but I am not relying on it, so I leave that discussion aside.

been constructed of parts, but rather appears to have been constructed from a single large block of ice. The third wall is constructed from a single large block of cement.[43]

Note that there are two distinctions here. The first is between objects that were constructed from parts, regardless of whether those parts retain their individuality after the object is constructed (the brick and ice walls), and objects that were not constructed from parts at all (the cement wall). The second is between objects that were constructed from parts and whose parts retain their individuality (the brick wall) and objects that were constructed from parts but whose parts do not retain their individuality (the ice wall).

Note also that these distinctions can be applied to representational systems. Written English sentences are constructed from words, but those words are kept separate from each other (by leaving spaces between the words in a sentence) such that they retain their individuality after being brought together to form sentences.[44] Imagine on the other hand a system that uses large piles of sand to represent the weights of various objects, and which combines the piles of sand (i.e., into a single larger pile of sand) to represent the combined weight of the objects. This would be a representational system whose compound representations are constructed from parts, but where those parts do not retain their individuality once they are combined. Finally, consider the familiar red-yellow-green stop lights used at intersections. Each of these colors is an atomic representation within the system, and they do not combine at all; the system does not employ compound representations.

[43] It is important to imagine the cement wall having been poured all at once. One should not imagine it being poured in separate forms that are then joined together.

[44] Some might object to the example, since words can be individuated even when spaces are not left between them. This is true, but the example is intended to show that there are at least some systems in which representations are constructed from parts and the parts retain their individuality. Indeed, even when a written sentence has no spaces, there remains a canonical breakdown of the sentence into words.

The point I want to make here is that mental magnitudes are of the second type of representational system. They are formed by combining increments, but in such a way that the system is unable to recover the number of increments that are compounded to form a magnitude. Or in other words, the parts (increments) that are combined to form compound representations (magnitudes) do not retain their individuality once they are so combined.[45] Since the increments are also variable in size, this has the result that the system cannot distinguish between representations of nearby numbers. Thus, focusing on magnitudes as completed representations (ignoring the increments that were combined to form them) leads to the conclusion that they are approximate representations of number. So, for example, Laurence and Margolis (2008) write that mental magnitudes "are by their nature approximate and hence incapable of expressing a difference of exactly one" (935). And Spelke (2003) writes that the system of mental magnitudes "represents large numbers of objects or events as sets with cardinal values, … however, [it] fails to represent sets exactly… and therefore it fails to capture the numerical operations of adding or subtracting one" (299).

However, we may also focus on magnitudes as representations that are formed by combining parts (increments) but whose parts do not retain their individuality after being combined. And focusing on them in this way yields a different result—it yields the result that mental magnitudes are formed by the composition of a precise number of increments. For recall that experimental studies have controlled for variables besides cardinality (such as total surface area or total brightness of a display) showing that what the system responds to is the number of

[45] It is important here to distinguish between two hypotheses. The first is that the accumulator encodes an increment for each object observed, and then combines these representations. In other words, it arrives at a completed representation of a group of objects by first forming a representation of each object in the group. The second is that the accumulator responds to individual objects, but does not encode separate representations for each – it creates the completed representation "all at once". I am here suggesting the second, and not the first of these hypotheses, at least in cases in which the objects are observed rapidly. The hypothesis I offer below does, however, depend on the accumulator being able to add increments to a magnitude that has already been encoded, when for example, the objects are observed slowly. I will say more about this distinction below.

objects or events in a group (i.e., the group's cardinality).[46] In other words, when presented with n objects, the system increments a magnitude n times. Of course, there is nothing approximate about the cardinality of a group of objects. There is nothing approximate about n, whatever n is.[47]

It is important to note here that the idea that magnitudes are composed of increments implies nothing at all about whether the medium of representation is continuous or discrete. For instance, the increments employed by the system described above that uses sand to represent weight are amounts of sand, a discrete medium. In contrast, a system that compounded cups of water would also employ increments, although the medium would be continuous (or at least, would appear continuous to the naked human eye). Indeed, as Dehaene describes the water-based accumulator, water is directed into the basin for a certain amount of time per object observed. This system also employs increments, even if the user does not re-direct the flow of water outside the basin between increments. An increment is just the amount of water directed into the basin during the specified period of time.

Now recall the necessary condition on possession of the concept *N* that I gave above: one must be able to distinguish groups of n objects from groups of n-1 objects and groups of n+1 objects. The worry that many authors seem to have is that, because the system of mental magnitudes does not distinguish a number from its nearest neighbors, it therefore cannot underlie this necessary condition on possession of natural number concepts. The point I am emphasizing now is that it is only when we focus on the magnitudes as completed representations that the

[46] As I noted above, some evidence suggests that the accumulator does not in fact respond to cardinality at all (at least in the small number range), but rather to continuous variables such as total contour length of the objects in a display or the density of the objects. Thus, one should read the proposal here as hypothetical, depending on the assumption that the accumulator creates a representation by incrementing a magnitude n times for n objects.
[47] Another way to make the point is that, while the literature on mental magnitudes has drawn a two-way distinction between cardinality represented and magnitude doing the representing, what is needed is a three-way distinction between cardinality represented, increments combined to form a total magnitude, and total magnitude.

system does not distinguish a number from its nearest neighbors. When we focus on them as representations that are formed by combining parts, then the system does indeed distinguish between a number and its nearest neighbors. It does so in the sense that it combines *n* increments to represent *n* objects.

Acknowledging the distinction between a mental magnitude and the increments that were combined to form it allows, I think, for an account of their role in the acquisition of natural number concepts that has been overlooked in the literature. Specifically, I want to suggest the possibility that learning the number words and the counting routine allows the child to store and recover the precise number of increments that combine to form a mental magnitude. As I noted in the introduction, my intention is to explore this hypothesis as an alternative to those extant in the literature and not to argue that it or any other is best supported by all the available evidence. Below I will present some evidence in its favor, but I will also note evidence against it, and I will discuss some unresolved challenges that it faces.

Recall the example of using a supply of water to count some number of objects. The increments of water are variable and compound and there is no way to recover the number of increments that were used to compose a representation. Thus, there is no guarantee that when two sets of objects with equal cardinality are counted, the resulting representations will appear as representations of the same cardinality. And there is no guarantee that when two sets of objects with unequal (though nearby) cardinalities are counted, the resulting representations will not appear as representations of the same cardinality.

Suppose we add a collection of pebbles, however, such that for every time an increment of water is added, a pebble is placed in a bin. Because there is a one-to-one correspondence between the number of objects counted and the number of increments of water employed, and also between the number of increments of water employed and the number of pebbles used, there will therefore be a one-to-one correspondence between the number of objects counted and the number of pebbles used. If the user of this system noticed this latter correspondence, she would be in a position to use the pebbles to compare the cardinalities of groups of objects, instead of the water.

Of course, because the user of this system can recover the exact number of pebbles that were compounded to form a representation, the system guarantees (barring user error) that when two sets of objects with equal cardinality are counted, the resulting representations will appear to be representations of equal cardinality. And it guarantees (again barring user error) that when two sets of objects with unequal (though nearby) cardinality are counted, the resulting representations will not appear as representations of the same cardinality. The present suggestion, then, is that learning the counting words is akin to adding the pebbles in the above metaphor. It provides a way of recovering the precision that is present in the formation of mental magnitudes, which is no longer present after they are formed. If this is right, then a child learning to use the number words must map those words to the increments that are compounded to form a magnitude,[48] rather than to the final product, the resulting magnitude. How can this be achieved?

To answer this question, it is important to recall that the experiments that have revealed the existence of the system of mental magnitudes depend on subjects not using language to count the objects in a display. For adult subjects, this is achieved by forcing rapid estimation of cardinality (and sometimes including recital of words other than number).[49] Thus they force the

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[48] Notice that this is not the same problem as somehow mapping number words to overlapping ranges of mental magnitudes. I take it that problem is unsolvable (as strong nativists seem to agree) because it requires mapping a precise system of representation directly to an approximate system. The present problem requires mapping a precise system of representation to another precise system (the number of increments combined in the formation of a mental magnitude).

[49] See e.g., Mandler and Shebo (1982), Whalen, Gallistel, and Gelman (1999), and Cordes, et al. (2001).

rapid production of mental magnitudes. Suppose however, that mental magnitudes may be produced slowly, one increment at a time, as a group of objects is slowly observed. In this case, recital of each counting word may serve to mark in memory the addition of a new increment to a magnitude that is being formed.

For instance, imagine a child who has mastered the counting routine as a meaningless string of words, but who does not yet understand the meanings of those words. As she counts a group of objects, that is, as she recites the counting routine one word for each object, she also creates an increment for each object in the group, and these increments are compounded together, creating a final magnitude. Thus, there is a one-to-one correspondence between the number of words that have been recited and the number of increments that were compounded.

That the child be able to slowly attend to each object, reciting a counting word for each, is important here. For, as noted above (footnote 45) when rapidly observing a set of objects, the system may not encode an increment for each object and then combine them, as two separate actions. Rather, it may simply create and combine increments "all at once." In that case, the child would not have the opportunity to notice that a new increment was added for each object observed, and thus would not be in a position to note a one-to-one correspondence between number words and increments. In short, it is crucial for the present hypothesis that during number word learning, some instances of counting take place slowly, such that the system is forced to encode one increment at a time, even if it would not do so were the subject observing a set of objects more rapidly.

Thus, the present hypothesis is that learning the meaning of the number words is a process whereby the child maps those words to the individual increments that compound to form magnitudes. In other words, it is a process whereby the child learns to use the number words to mark in memory the number of times a mental magnitude has been incremented. The number words therefore become precise representations of precise numbers of objects or events, by recording the precise numbers of increments that compose to form completed mental magnitudes.[50]

One piece of evidence for this view is the length of time it takes children to learn the meaning of number words, even after having mastered the counting routine. Recall that the process takes between a year and eighteen months, and proceeds in recognizable stages: children first become 1-knowers, then 2-knowers, then 3-knowers, and finally, *n*-knowers. Now suppose that the present hypothesis is true, and imagine a 1-knower learning to use the word "two".

The hypothesis is that she must learn that "two" is the appropriate word when and only when a magnitude is composed of exactly two increments. Of course, she can only know that if she knows that she has added exactly one increment to a magnitude that previously consisted of exactly one increment. But in order to know that, she must be in a position to know that the magnitude had previously been composed of exactly one increment. To know that, she needs some way of recalling that a magnitude had been so composed. According to the present hypothesis though, this is exactly what learning the meaning of the word "one" implies. In short, the hypothesis predicts that children must understand "one" before they understand "two", which in fact they do.

The same argument can be made with respect to learning "three". On the present hypothesis one cannot learn the meaning of "three" before one learns the meaning of "two", because it requires being in a position to notice that exactly one increment has been added to a magnitude that was previously composed of exactly two increments, and one cannot be in that

[50] There is also some evidence that learning a system of number words may also help to remove some of the variability inherent in magnitude representations. See for example, Piazza and Izard (2009).

position unless one knows that the magnitude was previously composed of exactly two increments. In general then, the present hypothesis predicts that number words must be learned in order, as in fact they are, and therefore that the process should take considerable time, as in fact it does.[51] [52]

Some may wonder in addition if there are neuroscientific studies that support the present hypothesis. Indeed, there are many such studies concerning numerical cognition in humans, including in children and infants, and also in non-human animals. For example, neuroimaging studies have found that adults, children, and infants represent non-symbolic numerical information in similar locations.[53] Single-neuron studies have implicated both the prefrontal cortex and the posterior parietal cortex in the primate brain in numerical (and length) representation.[54] Other studies provide neural network models of numerical representation,[55] and still others address the connections that develop between areas of the brain that represent external numerical symbols, such as number words, body parts, gestures, and numerals.[56] [57] However, it is unclear that neuroscientific studies can yet distinguish between the present hypothesis and the others discussed here.

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[51] It is true of course that Spelke (2003) argues that the length of time it takes children to learn the meaning of number words is evidence for her view—that the system of object-files must also play a role in that learning. There are two issues that point to the present hypothesis though, over Spelke's. First, the present hypothesis is simpler, in that it does not require object-files, as does Spelke's account. Second, notice that Spelke's account does not predict that the number words must necessarily be learned in order. It allows that they are, and of course that is what one would expect. But her hypothesis is compatible with a child learning the number words out of order. The present hypothesis is not so compatible. Again, it requires that children first learn the meaning of "one", then "two," and so on. On the other hand however, this may also be a drawback for the present hypothesis. See the discussion of "Alex" the grey parrot, below.

[52] Of course, after about "three" or "four" children learn the meanings of the rest of the number words all at once, which requires an inductive inference of some kind. None of the extant theories—including the present one— has an explanation of this part of the process. It must be acknowledged though, that the fact that that inductive inference happens after about "three" or "four" rather than, for example, at "two" or "five" is perhaps better explained by Spelke's view, since that view relies on mapping number words to object-files, and that system has a set-size limit of about three or four.

[53] See for example Cantlon et al. (2006) and Izard, Dehaene, and Dehaene (2008).

[54] See for example, Tudusciuc and Nieder (2007, 2009).

[55] See for example, Dehaene and Changeaux (1993) and Verguts and Fias (2004).

[56] See De Cruz (2008) for discussion.

[57] See Nieder and Dehaene (2009), for example, for a review of neuroscientific studies on numerical cognition.

The reason is because current neuroscientific studies are primarily concerned to locate numerical processing in the brain, to describe the kind of information represented at varying locations, (e.g., continuous, discrete), and to identify similarities in numerical processing across different areas of the brain. But consider, for example, the difference between the present hypothesis and Spelke's. Both claim that acquisition of precise number concepts involves associating number words with mental magnitudes. Whereas the present hypothesis claims that number words are mapped to successive additions of increments of magnitudes, however, Spelke claims that number words are mapped simultaneously to mental magnitudes and to object-files.

Confirming the present hypothesis and disconfirming Spelke's, at the neural level, would require showing something like the following: that the neural activity underlying slow recitation of the number words is highly correlated with the neural activity underlying the slow production of mental magnitudes (if indeed they are sometimes produced slowly) and that the former is not highly correlated with deployment of object-files. Moreover, confirmation of the present hypothesis would require showing that such a correlation between the neural activity underlying use of the number words and neural activity underlying creation of mental magnitudes develops over time—in particular, that it develops during the time period in which children progress from not knowing the meaning of the number words, to being 1-knowers, to being *n*-knowers. To my knowledge, so far there are no neuroscientific studies—or even collections of such studies—that can offer that kind of detailed confirmation.[58]

[58] Note that in discussing the neuroscientific evidence, I am only arguing that it cannot distinguish between the present account and the others discussed here. I am not arguing that there is no evidence speaking against the theory that an accumulator mechanism is responsible for numerical competence in infants. As I noted above (footnote 24), there is much debate about whether an accumulator or some other kind of mechanism is responsible, and if it is an accumulator, exactly how it functions. But those debates remain unsettled, so it is worth exploring, on the assumption that an accumulator is responsible for infant numerical competence, the space of possible hypotheses concerning the acquisition of number concepts. The present argument is that one important hypothesis within that space has been overlooked.

6. Comparisons and Challenges

In this section I will argue that the present hypothesis is not equivalent to any of the others considered here, and I will also discuss several important objections and challenges facing the present hypothesis. To begin, recall the first hypothesis described above. That hypothesis, offered by Gallistel, Gelman, and Cordes (2006) and suggested by others as well, was that the system of mental magnitudes is homomorphic to the counting routine, that the child recognizes this, and is therefore able to map count words onto states of the accumulator. But this hypothesis was later abandoned, because the approximate nature of completed magnitudes means that there are no equal states of the accumulator onto which count words can map. As Leslie, Gelman, and Gallistel (2008) put the point,

> A fundamental intuition here is that if three things are counted, then the resulting cardinal value will be exactly equal to the cardinal value that will result from counting them again. It is hard to account for this intuition if the brain represents cardinal values by noisy reals because two exactly equal values will never occur." (213)

But this makes it clear that they are considering the words as being mapped to completed magnitudes, and not to the number of increments that composed to form those magnitudes. For it is only the completed magnitudes that are never equal. The number of increments will be equal, if the cardinality of the set being counted remains the same. Since the current hypothesis focuses on the number of increments composed, while the other hypothesis focuses on the completed magnitude, they are not after all the same hypothesis.

The present account also differs from the strongly nativist accounts presented above. After all, the hallmark of such accounts is the view that human beings possess innate natural number concepts, and in some sense the present account argues for that very claim. For on the present account, the process whereby mental magnitudes are formed—an innate process—itself contains the precision required to represent the natural numbers. However, both Leslie et al.'s

(2008) account and Margolis and Laurence's (2008) account[59] deny that the system of mental magnitudes can be the source of natural number concepts. To account for natural number concepts, they posit an innate system in addition to the system of mental magnitudes. The present account posits no such additional system. Rather, the present claim is that while the needed precision is contained in the process whereby increments are composed to form magnitudes, that precision cannot be exploited until after acquisition of the number words and the counting routine.

Nor indeed is the present account equivalent to Spelke's. For according to Spelke, children learn the meanings of number words by noticing that "the progression from [*n*] to [*n+1*] in the counting routine is marked by *the addition of one individual to the set* [and by] *an increase in the cardinal value of the set*" (302). The latter of these marks, however, is represented by the addition of an increment to a magnitude. Thus the view appears to be (in part) that children map number words to the addition of increments, as is the case according the view offered here. But Spelke also claims that object-files play a role, and indeed, that the number word "one" is initially mapped only to that system, and that other small number words are mapped both to object-files and incremented magnitudes. The appeal to both systems is, after all, what makes hers a weakly nativist view. But the present account makes no appeal to object-files. On this account, for example, children learn that when a magnitude is composed of exactly one increment, the number word "one" applies to that magnitude, that when a magnitude is composed of exactly two increments, the word "two" applies, and so on.

Finally, the present account differs significantly from Carey's as well. Recall that her account depends on the system of object-files, enriched by quantificational markers. It does not depend on the system of mental magnitudes at all (in the early stages). Since the present

[59] And Rips et al.'s (2008) account as well.

hypothesis does depend on mental magnitudes, and does not depend on object-files or quantificational markers, the accounts are not equivalent. Still, some might object that the present account does not after all depend on mental magnitudes, but instead merely on the ability to create one-to-one correspondences. It is true that this ability plays an important role in the present account (see footnote 3), and it is also true that completed mental magnitudes do not play an important role in the present account, as they do not in Carey's. But still, the operation of the system of mental magnitudes does play a central role, since according to the account number words are mapped to the individual increments that are tokened and compounded to form completed magnitudes.[60]

Granting that the present hypothesis is distinct from others in the literature, there are nevertheless important objections and challenges that it faces. Here I'll present four such concerns. First, some may question whether the number of increments in a magnitude are open to awareness in a way that would enable children to notice that there is, for example, a one-to-one correspondence between the number of increments added to a magnitude on a given occasion of counting and the number words recited on that same occasion. It is important to note though, that the present hypothesis does not claim that children are consciously aware of any part of the process in question, beyond their recitation of the number words and their attending to objects or events. Mental magnitudes themselves are not available for conscious inspection—for children or adults—and no one has proposed a theory that assumes they are. In other words, if mental

[60] Famously, Frege (1884/1953) showed that integer concepts could be derived from one-to-one correspondence, which in turn could be reduced to logical relationships alone. I noted above that all of the accounts described here will have to rely on the ability to create one-to-one correspondences. But they all also rely on innate representational systems—mental magnitudes, object-files, or innate representations of natural numbers—and so are all distinct from Frege's view.

magnitudes play a role in the acquisition of precise number concepts, that role is not open to introspection.

Second, I argued above that the length of time it takes children to learn the meaning of number words, even after having mastered the counting routine, is evidence for the present hypothesis. Indeed, I argued that the present hypothesis implies that children must learn the meanings of the number words in order, passing through the *n*-knower stages as they do. However, this may be a drawback for the present hypothesis. Irene Pepperberg and her colleagues trained “Alex”, a grey parrot, to correctly use the number words “one” through “six”. Alex did not learn the words in order though, thus showing that it is possible to learn these meanings out of order. Of course if learning the number words out of order is possible for a grey parrot, one might expect it to be possible for human children as well, which indeed would be problematic and perhaps fatal for the present hypothesis. Still, no examples of children learning the number words out of order have been given. Moreover, Alex’s training was importantly different than the process by which children learn the number words, as he was never taught to recite the count list. Thus I take the Alex studies to be, so far, inconclusive with respect to the possibility of children learning the meaning of the number words out of order, given knowledge of the counting routine (see Pepperberg 1994, Pepperberg and Gordon 2005, and Pepperberg and Carey 2012).

Third, in her 2009 book *The Origin of Concepts,* Susan Carey considers what she calls “Proposal 1”, which she describes as the idea that “analog magnitude representations are the numerical foundation for numeral list representations of number” (2009b, 309), and she provides evidence that she takes to “conclusively rule out” Proposal 1. Primarily that evidence is based on data from LeCorre and Carey (2006), which suggest that children are unable to estimate (verbally) the number of objects in a group until about six months after they become *n*-knowers (that is, after they understand the meaning of *n*, for any *n*).

In particular, LeCorre and Carey showed *n*-knowers cards with between one and ten dots. The cards were not displayed long enough for the children to count, but the children were asked to say how many dots were on the cards. One group of these children was successful at this task, and another was not. In short, although the unsuccessful children were *n*-knowers, they were unable to token a magnitude and then translate that magnitude into a corresponding number word. They had not, that is, mapped number words to completed magnitudes. Since they knew the meanings of the number words however, LeCorre and Carey conclude that mapping magnitudes to number words was not part of the process by which they learned the meanings of those words. Moreover, the group that was successful had an average age of about six months older than the average age of the unsuccessful group, and LeCorre and Carey therefore concluded that magnitudes are not mapped to number words until about six months after children become *n*-knowers.

This argument does seem fatal for Proposal 1, but Proposal 1 is the early proposal that was considered above, that children map number words directly to completed mental magnitudes, and that hypothesis is distinct from the present hypothesis, such that it is not clear that the argument is also fatal for the present hypothesis. The present hypothesis is that children map number words to the (precise number of) increments that are compounded to create magnitudes, not to the completed magnitudes themselves. It requires that children sometimes slowly increment a magnitude as they slowly count objects. It does not imply that children be able to create a magnitude as an estimate of the number of objects in a group that was observed rapidly and then produce a number word based on that magnitude. The present hypothesis is silent as to when children should be able to do that. That is, it is silent as to when children should have mapped number words to completed mental magnitudes. Thus it allows that even after children have learned the meanings of the number words, by mapping those words to the increments that compose magnitudes, they may nevertheless not have mapped number words to completed magnitudes. Thus LeCorre and Carey's data does not seem fatal for the present proposal, as it does for Proposal 1.

Still, it might seem odd that children would be able to use number words to store and recall the number of increments composed to form a magnitude, yet not have mapped number words to completed magnitudes. So while Carey's argument is not clearly fatal for the present hypothesis, it does point to this tension. Moreover, it points to the fact that the present hypothesis has no answer to the question when children should have mapped number words to completed magnitudes. Relieving that tension and answering that question remain open challenges for the present hypothesis.

Finally, I noted above (footnote 24) that there is debate about whether the system of mental magnitudes operates serially or in parallel. That is, when a subject observes a group of objects, does the system compose all the increments at once, or does it compose them one at a time? It may seem that the present account depends on the system operating serially, since it depends on the child learning to map each successive count word to the addition of a new increment to a magnitude. If all the increments were added at once, it would be impossible to notice each new addition of an increment, so as to mark it with a number word. Indeed, the present account does depend on the possibility that the system be able to operate serially. In particular, the present account depends on the subject observing the objects slowly enough to count them. In that case, the system needs to be able to operate serially, incrementing as the subject observes each object.

Of course, that the system operates when objects or events are observed in succession—even if too rapidly to count—provides evidence that the system operates serially. For it would seem unlikely that the system wait until the subject has seen all the objects or events she will see, and only then compound the increments all at once. The system cannot know, after all, when the subject had seen all the objects she will see. It seems much more likely that the system would add increments to a magnitude as objects are observed. If they are observed in succession, then the increments would be added serially. And if they are observed in succession rather slowly, then the increments would be added serially, and slowly enough for the subject to notice the addition of each new increment. This does not rule out the possibility that the system operate in parallel though, when a set of objects are observed all at once. Thus, while the present hypothesis depends on the system being able to operate serially, it does not depend on it only operating serially.[61] [62] Still, this points to another area in which the present hypothesis needs attention. Since the empirical evidence is largely based on rapidly observed objects and events, more confirmation of the idea that the accumulator functions serially when objects or events are observed slowly is needed.

Conclusion

Thus I have argued that there is an important distinction between a mental magnitude and the increments that were composed to form it. While completed magnitudes are approximate

[61] Still, according the present hypothesis, instances in which the system operates in parallel cannot play a role in the acquisition of natural number concepts. Only instances in which the system operates serially can.

[62] There are, however, explanations of how it would be possible to create magnitude representations in parallel even when objects are presented serially (see e.g., Church and Broadbent 1990).

representations of number, they are composed by a precise number of increments. I believe this distinction has largely been overlooked in literature on mental magnitudes, and therefore so too has an important hypothesis concerning the role of mental magnitudes in the acquisition of precise number concepts.

That hypothesis is that acquisition of the number words and the counting routine allows the child to use that routine, under ideal circumstances, to mark in memory the number of increments that were composed to form a magnitude. Since a magnitude represents n objects or events by composing n increments, and because n is precise, recitation of the counting routine therefore serves to store in memory a precise representation of the number of objects or events that were observed. This hypothesis differs from strong nativist views in that it does not posit innate natural number concepts. It differs from weak nativist views in that it claims that the system of mental magnitudes is the only innate system to play a role in the acquisition of natural number concepts. This hypothesis does face certain challenges, but as other extant theories do as well, exploring alternative hypotheses remains a useful endeavor.

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